Lesson 6.4 Exercises, pages 478–489

3. a) For each triangle, write the Sine Law equation you would use to determine the length of AC.

i) \[ \angle B = 30^\circ, \angle C = 80^\circ, c = 5 \]

\[ \frac{b}{\sin 30^\circ} = \frac{5}{\sin 80^\circ} \]

ii) \[ \angle B = 40^\circ, \angle C = 110^\circ, c = 10 \]

\[ \frac{b}{\sin 40^\circ} = \frac{10}{\sin 110^\circ} \]
b) For each triangle in part a, determine the length of AC to the nearest tenth of a centimetre.

i) \( b = \frac{5 \sin 30\degree}{\sin 80\degree} \)
\[ b = 2.5385 \ldots \]
AC is approximately 2.5 cm.

ii) \( b = \frac{10 \sin 40\degree}{\sin 110\degree} \)
\[ b = 6.8404 \ldots \]
AC is approximately 6.8 cm.

4. a) For each triangle, write the Sine Law equation you would use to determine the measure of \( \angle E \).

i) 
\[ \triangle DEF \]
\[ \begin{align*}
\text{Use:} & \quad \frac{\sin E}{d} = \frac{\sin D}{e} \\
\text{Substitute:} & \quad \angle D = 70\degree, e = 8, d = 10 \\
\sin E &= \frac{\sin 70\degree}{10} \\
\angle E &= 48.7425 \ldots \degree
\end{align*} \]

ii) 
\[ \triangle DEF \]
\[ \begin{align*}
\text{Use:} & \quad \frac{\sin E}{f} = \frac{\sin F}{e} \\
\text{Substitute:} & \quad \angle F = 115\degree, e = 7, f = 10 \\
\sin E &= \frac{\sin 115\degree}{10} \\
\angle E &= 39.3766 \ldots \degree
\end{align*} \]

b) For each triangle in part a, determine the measure of \( \angle E \) to the nearest degree.

i) \( \angle E = \sin^{-1} \left( \frac{8 \sin 70\degree}{10} \right) \)
\[ \angle E = 48.7425 \ldots \degree \]
\[ \angle E = 49\degree \]

ii) \( \angle E = \sin^{-1} \left( \frac{7 \sin 115\degree}{10} \right) \)
\[ \angle E = 39.3766 \ldots \degree \]
\[ \angle E = 39\degree \]
5. For each triangle, determine the measure of $\angle J$ to the nearest degree.

a) $G \quad 9 \text{ cm} \quad 50^\circ \quad H \quad 7 \text{ cm} \quad J$

Use: $\sin \frac{J}{j} = \frac{\sin H}{h}$

Substitute:

$\angle H = 50^\circ, j = 9, h = 7$

$\sin J = \frac{9 \sin 50^\circ}{7}$

$\sin J = \frac{9 \sin 50^\circ}{7}$

$\sin^{-1} \left( \frac{9 \sin 50^\circ}{7} \right) = 80^\circ$

Since $\angle J$ is obtuse:

$\angle J = 180^\circ - 80^\circ, \text{ or } 100^\circ$

b) $G \quad 15 \text{ cm} \quad 50^\circ \quad H \quad 12 \text{ cm} \quad J$

Use: $\sin \frac{J}{j} = \frac{\sin H}{h}$

Substitute:

$\angle H = 50^\circ, j = 15, h = 12$

$\sin J = \frac{15 \sin 50^\circ}{12}$

$\sin J = \frac{15 \sin 50^\circ}{12}$

$\sin^{-1} \left( \frac{15 \sin 50^\circ}{12} \right) = 73^\circ$

Since $\angle J$ is obtuse:

$\angle J = 180^\circ - 73^\circ, \text{ or } 107^\circ$

6. Given the following information about each possible $\triangle ABC$, determine how many triangles can be constructed.

a) $c = 10 \text{ cm}, a = 12 \text{ cm}, \angle A = 20^\circ$

The ratio of the side opposite the angle to the side adjacent to the angle is: $\frac{a}{c} = \frac{12}{10}$, which is 1.2

Since $\frac{a}{c} > 1$, only 1 triangle can be constructed

b) $c = 18 \text{ cm}, a = 12 \text{ cm}, \angle A = 20^\circ$

The ratio of the side opposite the angle to the side adjacent to the angle is: $\frac{a}{c} = \frac{12}{18}$, which is 0.6

$\sin 20^\circ = 0.3420...

Since \sin 20^\circ < 0.6 < 1$, two triangles can be constructed

c) $c = 18 \text{ cm}, a = 12 \text{ cm}, \angle A = 50^\circ$

The ratio of the side opposite the angle to the side adjacent to the angle is: $\frac{a}{c} = \frac{12}{18}$, which is 0.6

$\sin 50^\circ = 0.7660...

Since $\frac{a}{c} < \sin 50^\circ$, no triangle can be constructed
7. a) For each triangle, determine the length of MN to the nearest tenth of a centimetre.

i) Determine \( \angle N \).

Use: \( \frac{\sin N}{n} = \frac{\sin M}{m} \)

Substitute: \( \angle M = 100^\circ \),
\( n = 9, m = 13 \)
\( \sin N = \frac{9 \sin 100^\circ}{13} \)
\( \sin N = \frac{9 \sin 100^\circ}{13} \)

Since \( \angle N \) is acute:
\( \angle N = \sin^{-1} \left( \frac{9 \sin 100^\circ}{13} \right) \)
\( \angle N = 42.9836...^\circ \)

So, \( \angle K \)
\( = 180^\circ - (100^\circ + 42.9836...^\circ) \)
\( = 37.0163...^\circ \)

Use: \( \frac{k}{\sin K} = \frac{m}{\sin M} \)

Substitute: \( \angle K = 37.0163...^\circ \)
\( \angle M = 100^\circ, m = 13 \)
\( \frac{k}{\sin 37.0163...^\circ} = \frac{13}{\sin 100^\circ} \)
\( k = \frac{13 \sin 37.0163...^\circ}{\sin 100^\circ} \)
\( k = 7.9472... \)

MN is approximately 7.9 cm.

ii) Determine \( \angle N \).

Use: \( \frac{\sin N}{n} = \frac{\sin M}{m} \)

Substitute: \( \angle M = 40^\circ \),
\( n = 18, m = 15 \)
\( \sin N = \frac{18 \sin 40^\circ}{15} \)
\( \sin N = \frac{18 \sin 40^\circ}{15} \)

Since \( \angle N \) is acute:
\( \angle N = \sin^{-1} \left( \frac{18 \sin 40^\circ}{15} \right) \)
\( \angle N = 50.4785...^\circ \)

So, \( \angle K \)
\( = 180^\circ - (40^\circ + 50.4785...^\circ) \)
\( = 89.5215...^\circ \)

Use: \( \frac{k}{\sin K} = \frac{m}{\sin M} \)

Substitute: \( \angle K = 89.5215...^\circ \)
\( \angle M = 40^\circ, m = 15 \)
\( \frac{k}{\sin 89.5215...^\circ} = \frac{15}{\sin 40^\circ} \)
\( k = \frac{15 \sin 89.5215...^\circ}{\sin 40^\circ} \)
\( k = 23.365... \)

MN is approximately 23.3 cm.

b) Suppose you had been given the measures for the triangles in part a and not the diagrams. In which triangle would there have been an ambiguous case? Justify your choice.

i) Angle M is obtuse, so there is only 1 triangle

ii) The ratio of the side opposite the angle to the side adjacent to the angle is:
\( \frac{NK}{KM} = \frac{15}{18} \) which is 0.83
\( \sin 40^\circ = 0.6427... \)

Since \( \sin 40^\circ < 0.83 < 1 \), then there is an ambiguous case.
8. Here is a part of a proof of the Sine Law for \( \triangle UVW \) with obtuse \( \angle W \).

Explain each step.

In \( \triangle VUT \),
\[
\sin U = \frac{h}{w} \quad \text{Using the sine ratio in a right triangle}
\]
Solving for \( h \)

In \( \triangle VWT \),
\[
\sin \beta = \frac{h}{w} \quad \text{Using the sine ratio in a right triangle}
\]
Supplementary angles on a line
\[
\sin \beta = \sin (180^\circ - \alpha) \quad \text{The sine of an angle is equal to the sine of its supplement.}
\]
So, \( \sin W = \frac{h}{u} \)
\[
\frac{h}{u} = \frac{u \sin W}{w} \quad \text{Solving for } h
\]
Equating expressions for \( h \)
\[
\frac{u \sin W}{w} = \frac{w \sin U}{u} \quad \text{Dividing each side by } \sin W \sin U
\]

9. Solve each triangle. Give the angle measures to the nearest degree and side lengths to the nearest tenth of a centimetre.

\[ a) \]

\begin{align*}
\text{Determine } \angle T. \\
\text{Use: } \sin \frac{T}{s} = \frac{s}{t} \\
\text{Substitute: } \angle S = 35^\circ, \\
t = 24, s = 19 \\
\sin \frac{T}{24} = \sin \frac{35^\circ}{19} \\
\sin \frac{T}{24} = \sin \frac{35^\circ}{19} \\
\sin T = \frac{24 \sin 35^\circ}{19} \\
\text{Since } \angle T \text{ is acute:} \\
\angle T = \sin^{-1} \left( \frac{24 \sin 35^\circ}{19} \right) \\
\angle T = 46.4287...^\circ \\
\angle T \approx 46^\circ, \quad \angle R \approx 99^\circ, \\
ST \approx 32.8 \text{ cm}
\end{align*}
b) In $\triangle KMN$, $\angle M = 70^\circ$, $KN = 14.1$ cm, and $MK = 14.5$ cm

Check how many triangles can be drawn.

The ratio of the side opposite $\angle M$ to the side adjacent to $\angle M$ is:

$$\frac{KN}{KM} = \frac{14.1}{14.5}$$

which is 0.9724...

Since $\sin 70^\circ < 0.9724... < 1$,
two triangles can be constructed:

$\triangle KMN_1$ is acute; $\triangle KMN_2$ is obtuse.

In $\triangle KMN_1$

Determine $\angle N_1$.

Use: $\sin N_1 = \frac{m}{n}$

Substitute: $\angle M = 70^\circ$

$n = 14.5, m = 14.1$

$\sin N_1 = \frac{14.1}{14.5}$

$\angle N_1 = \sin^{-1} \left( \frac{14.1}{14.5} \sin 70^\circ \right)$

$\angle N_1 = 75.0943^\circ$

So, $\angle K$

$= 180^\circ - (70^\circ + 75.0943^\circ)$

$= 34.9056^\circ$

Use: $\frac{k}{\sin K} = \frac{m}{\sin M}$

Substitute: $\angle K = 34.9056^\circ$

$\angle M = 70^\circ, m = 14.1$

$k = \frac{14.1}{\sin 34.9056^\circ}$

$k = \frac{14.1}{\sin 70^\circ}$

$k = 8.5862...$

$\angle N_1 = 75^\circ$, $\angle K = 35^\circ$

$MN_1 = 8.6$ cm

In $\triangle KMN_2$

Determine $\angle N_2$.

$\angle N_2 = 180^\circ - 75.0943^\circ$

$= 104.9056^\circ$

So, $\angle K$

$= 180^\circ - (70^\circ + 104.9056^\circ)$

$= 5.0943^\circ$

Use: $\frac{k}{\sin K} = \frac{m}{\sin M}$

Substitute: $\angle K = 5.0943^\circ$

$\angle M = 70^\circ, m = 14.1$

$k = \frac{14.1}{\sin 5.0943^\circ}$

$k = \frac{14.1}{\sin 70^\circ}$

$k = 1.3323...$

$\angle N_2 = 105^\circ$, $\angle K = 5^\circ$

$MN_2 = 1.3$ cm
10. For each triangle below, can you use the Sine Law to determine the indicated measure? If your answer is yes, determine the measure to the nearest tenth of a unit. If your answer is no, explain why.

a) No, because only one angle is given and it is a contained angle.

b) No, because no angles are given.

c) Yes; in \( \triangle GHJ \),
\[ \angle H = 180^\circ - 110^\circ = 70^\circ \]
Use: \[ \frac{h}{\sin H} = \frac{g}{\sin G} \]
Substitute: \( \angle H = 70^\circ \),
\[ \angle G = 30^\circ; g = 10 \]
\[ \frac{h}{\sin 70^\circ} = \frac{10}{\sin 30^\circ} \]
\[ h = \frac{10 \sin 70^\circ}{\sin 30^\circ} = 18.7938\ldots \]
\[ h \approx 18.8 \text{ cm} \]

d) Yes
Use: \( \frac{\sin N}{n} = \frac{\sin K}{k} \)
Substitute: \( \angle K = 48^\circ \),
\[ n = 11, k = 10 \]
\[ \sin N = \frac{11 \sin 48^\circ}{10} \]
\[ \sin N = 54.8312\ldots^\circ \]
Since \( \angle N \) is acute:
\[ \angle N = \sin^{-1}\left(\frac{11 \sin 48^\circ}{10}\right) \]
\[ \angle N = 54.8312\ldots^\circ \]
\[ \theta = 180^\circ - (48^\circ + 54.8312\ldots^\circ) \]
\[ \theta = 77.1687\ldots^\circ \]
\[ \theta = 77.2^\circ \]
11. A surveyor constructed this drawing of a triangular lot.

a) Determine the unknown side lengths.

\[ \angle M = 180^\circ - (25^\circ + 110^\circ) = 45^\circ \]

Use: \( \frac{p}{\sin P} = \frac{m}{\sin M} \)

Substitute: \( \angle P = 110^\circ, \angle M = 45^\circ, m = 270 \)

\[ p = \frac{270 \sin 110^\circ}{\sin 45^\circ} \approx 358.8 \text{ m} \]

MN is approximately 359 m.

b) Determine the total length of fencing needed to enclose the lot.

PM is approximately 161 m.

The total length of fencing is:

\[ 359 \text{ m} + 161 \text{ m} + 270 \text{ m} = 790 \text{ m} \]

12. a) Solve \( \triangle PQR \) by drawing a perpendicular from P to QR, then use primary trigonometric ratios in each right triangle formed.

Give the angle measures to the nearest degree and the side lengths to the nearest tenth of a centimetre.

In \( \triangle PRN \),

\[ \cos 74^\circ = \frac{RN}{5} \]

\[ RN = 5 \cos 74^\circ, \quad RN = 1.378 \ldots \]

\[ \sin 74^\circ = \frac{PN}{5} \]

\[ PN = 5 \sin 74^\circ, \quad PN = 4.8063 \ldots \]

In \( \triangle PQN \),

\[ \sin Q = \frac{4.806 \ldots}{6.5} \]

\[ Q = 47.683 \ldots^\circ \]

In \( \triangle PQR \),

\[ \angle P = 180^\circ - (48^\circ + 74^\circ) = 58^\circ \]

\[ RQ = (1.378 \ldots + 4.376 \ldots) \text{ cm} = 5.8 \text{ cm} \]
b) Use the Sine Law to solve \( \triangle PQR \).

In \( \triangle PQR \), use: 
\[
\frac{\sin Q}{q} = \frac{\sin R}{r}
\]
\[
\frac{\sin Q}{5} = \frac{\sin 74^\circ}{6.5}
\]
\[
\sin Q = \frac{5 \sin 74^\circ}{6.5}
\]
Since \( \angle Q \) is acute:
\[
\angle Q = \sin^{-1}\left(\frac{5 \sin 74^\circ}{6.5}\right)
\]
\[
\angle Q = 47.6830\ldots^\circ
\]
\[
\angle P = 180^\circ - (47.6830\ldots^\circ + 74^\circ)
\]
\[
\angle P = 58.3169\ldots^\circ
\]
Use:
\[
\frac{\sin P}{p} = \frac{\sin R}{r}
\]
\[
p = \frac{6.5 \sin 58.3169\ldots^\circ}{\sin 74^\circ}
\]
\[
p = 5.7541\ldots
\]
\[
\angle Q = 48^\circ, \angle P = 58^\circ, RQ = 5.8 \text{ cm}
\]

c) Which strategy for solving \( \triangle PQR \) was more efficient? Justify your answer.

The Sine Law needed 3 calculations, while the primary trigonometric ratios needed 6 calculations; so the Sine Law is more efficient.

13. A sailor made this sketch on her navigation chart. How much closer is the ship at \( S \) to lighthouse \( A \) than to lighthouse \( B \)?

In \( \triangle SAB \),
\[
\angle ASB = 90^\circ - (10^\circ + 20^\circ)
\]
\[
= 60^\circ
\]
So, \( \angle SAB = 180^\circ - (60^\circ + 35^\circ) \)
\[
= 85^\circ
\]
Use: 
\[
\frac{b}{\sin B} = \frac{s}{\sin S}
\]
Substitute: \( \angle B = 35^\circ \), 
\[
\angle S = 60^\circ, s = 11
\]
\[
\frac{b}{\sin 35^\circ} = \frac{11}{\sin 60^\circ}
\]
\[
b = \frac{11 \sin 35^\circ}{\sin 60^\circ}
\]
\[
b = 7.2853\ldots
\]
The required distance is: 12.6533\ldots - 7.2853\ldots = 5.3679\ldots
The ship is approximately 5 km closer to lighthouse \( A \).
14. Two ships are 1600 m apart. Each ship detects a wreck on the ocean floor. The wreck is vertically below the line through the ships. From the ships, the angles of depression to the wreck are 40° and 28°.

a) To the nearest metre, how far is the wreck from each ship?

The wreck could be between the ships or on one side of both ships. Sketch a diagram to show both positions.

\[ \angle W_1 = 180^\circ - (28^\circ + 40^\circ) = 112^\circ \]

\[ \frac{a}{\sin A} = \frac{w}{\sin W_1} \]

Substitute: \( \angle A = 28^\circ \)

\[ \angle W_1 = 112^\circ; w = 1600 \]

\[ a = \frac{1600 \sin 28^\circ}{\sin 112^\circ} \]

\[ a = 810.1462\ldots \]

Use: \( b = \frac{1600 \sin 40^\circ}{\sin 112^\circ} \)

Substitute: \( \angle B = 40^\circ \)

\[ b = 1109.2300\ldots \]

The wreck is approximately 810 m and 1109 m from the ships.

b) To the nearest metre, what is the depth of the wreck?

In \( \triangle ABW_1 \), draw the perpendicular from \( W_1 \) to \( AB \) at \( N \).

Then, \( W,N \) is the depth of the wreck.

In right \( \triangle ANW_1 \):

\[ \sin 28^\circ = \frac{W,N}{\sin W_1} \]

\[ W,N = 1109.2300\ldots (\sin 28^\circ) \]

\[ W,N = 520.7519\ldots \]

The wreck is approximately 521 m deep or 2322 m deep.

In \( \triangle ABW_2 \), draw the perpendicular from \( W_2 \) to \( AB \) extended to \( M \).

Then, \( W,M \) is the depth of the wreck.

In right \( \triangle AMW_2 \):

\[ \sin 28^\circ = \frac{W,M}{\sin W_2} \]

\[ W,M = 4946.6202\ldots (\sin 28^\circ) \]

\[ W,M = 2322.2975\ldots \]

The wreck is approximately 3613 m and 4947 m from the ships.
15. Two angles in a triangle measure 60° and 45°. The longest side is 10 cm longer than the shortest side. Determine the perimeter of the triangle to the nearest tenth of a centimetre.

Sketch a diagram. There are 2 solutions.

In $\triangle ABC_1$,
$\angle A = 180^\circ - (38.0^\circ + 57.8^\circ) = 84.2^\circ$

Use: $\frac{a}{\sin A} = \frac{c}{\sin C_1}$

Substitute: $\angle A = 84.2^\circ$, $\angle C_1 = 57.8^\circ$, $c = 19.8$

\[
a = \frac{19.8 \sin 84.2^\circ}{\sin 57.8^\circ} \\
a = 23.2791\ldots
\]

The people are approximately 23.3 m or 7.9 m apart.

16. Two angles in a triangle measure 60° and 45°. The longest side is 10 cm longer than the shortest side. Determine the perimeter of the triangle to the nearest tenth of a centimetre.

Sketch a diagram.
The third angle in the triangle is: $180^\circ - (45^\circ + 60^\circ) = 75^\circ$
The shortest side is opposite the least angle, so let $BC = a$.
The longest side is opposite the greatest angle, so $AB = a + 10$.

Use: $\frac{a}{\sin A} = \frac{c}{\sin C}$

Substitute: $\angle A = 45^\circ$, $\angle C = 75^\circ$, $c = a + 10$

\[
a = \frac{a + 10}{\sin 45^\circ} \\
a \sin 75^\circ = a \sin 45^\circ + 10 \sin 45^\circ \\
a(\sin 75^\circ - \sin 45^\circ) = 10 \sin 45^\circ \\
a = \frac{10 \sin 45^\circ}{\sin 75^\circ - \sin 45^\circ} \\
a = 27.3205\ldots
\]

Use: $\frac{b}{\sin B} = \frac{a}{\sin A}$

Substitute: $\angle B = 60^\circ$, $\angle A = 45^\circ$, $a = 27.3205\ldots$

\[
b = \frac{27.3205\ldots \sin 60^\circ}{\sin 45^\circ} \\
b = 27.3205\ldots \sin 60^\circ \\
b = 33.4606\ldots
\]

The perimeter is: $(27.3205\ldots + 27.3205\ldots + 10 + 33.4606\ldots) \text{ cm} = 98.1 \text{ cm}$
17. A hiker plans a trip in two sections. Her destination is 15 km away on a bearing of N70°E from her starting position. The first leg of the trip is on a bearing of N10°E. The second leg of the trip is 14 km. How long is the first leg? Give the answer to the nearest tenth of a kilometre.

Sketch a diagram.
∠DCB = 70° – 10°
= 60°

Check how many triangles can be drawn. The ratio of the side opposite ∠C to the side adjacent to ∠C is:
\[
\frac{BD}{BC} = \frac{14}{15}, \text{which is 0.93}
\]
\[
\sin 60° = 0.8660\ldots
\]
Since \(\sin 60° < 0.93 < 1\), two triangles can be constructed:
\(\angle DCB\) is acute; \(\angle DCB\) is obtuse

In \(\triangle D_1CB\)
Determine \(\angle D_1\).
Use: \[
\frac{\sin D_1}{d} = \frac{\sin C}{c}
\]
Substitute: \(\angle C = 60°\), \(d = 15\), \(c = 14\)
\[
\frac{\sin D_1}{15} = \frac{\sin 60°}{14}
\]
\[
\sin D_1 = \frac{15 \sin 60°}{14}
\]
\[
\angle D_1 = 68.1073\ldots°
\]
\[
\angle B = 180° - (60° + 68.1073\ldots°)
\]
\[
= 51.8926\ldots°
\]
To determine \(CD_1\),
use: \[
\frac{b}{\sin B} = \frac{c}{\sin C}
\]
Substitute: \(\angle B = 51.8926\ldots°\), \(\angle C = 60°\), \(c = 14\)
\[
\frac{b}{\sin 51.8926\ldots°} = \frac{14}{\sin 60°}
\]
\[
b = \frac{14 \sin 51.8926\ldots°}{\sin 60°}
\]
\[
b = 12.7201\ldots
\]
The first leg is approximately 12.7 km or 2.3 km.

In \(\triangle D_2CB\)
So, \(\angle B = 180° - (60° + 111.8926\ldots°)\)
\[
= 8.1073\ldots°
\]
To determine \(CD_2\),
use: \[
\frac{b}{\sin B} = \frac{c}{\sin C}
\]
Substitute: \(\angle B = 8.1073\ldots°\), \(\angle C = 60°\), \(c = 14\)
\[
\frac{b}{\sin 8.1073\ldots°} = \frac{14}{\sin 60°}
\]
\[
b = \frac{14 \sin 8.1073\ldots°}{\sin 60°}
\]
\[
b = 2.2798\ldots
\]