Lesson 4.3 Exercises, pages 298–304

4. Use these tables to determine each value below.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>x</th>
<th>g(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>-4</td>
<td>-3</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
<td>4</td>
<td>13</td>
</tr>
</tbody>
</table>

a) \(f(g(-2))\)

From the 2nd table: \(g(-2) = 1\)
From the 1st table: \(f(1) = -2\)
So, \(f(g(-2)) = -2\)

b) \(g(f(-2))\)

From the 1st table: \(f(-2) = 4\)
From the 2nd table: \(g(4) = 13\)
So, \(g(f(-2)) = 13\)

c) \(f(f(-1))\)

From the 1st table: \(f(-1) = 2\)
From the 1st table: \(f(2) = -4\)
So, \(f(f(-1)) = -4\)

d) \(g(f(0))\)

From the 1st table: \(f(0) = 0\)
From the 2nd table: \(g(0) = 5\)
So, \(g(f(0)) = 5\)
5. Given the graphs of \( y = f(x) \) and \( y = g(x) \), determine each value below.

a) \( f(g(-1)) \)

From the graph of \( y = g(x) \), \( g(-1) = 1 \)
From the graph of \( y = f(x) \), \( f(1) = -2 \)
So, \( f(g(-1)) = -2 \)

b) \( g(f(-2)) \)

From the graph of \( y = f(x) \), \( f(-2) = 1 \)
From the graph of \( y = g(x) \), \( g(1) = 5 \)
So, \( g(f(-2)) = 5 \)

c) \( g(g(-2)) \)

From the graph of \( y = g(x) \), \( g(-2) = -1 \)
From the graph of \( y = g(x) \), \( g(-1) = 1 \)
So, \( g(g(-2)) = 1 \)

d) \( f(g(1)) \)

From the graph of \( y = g(x) \), \( g(1) = 5 \)
From the graph of \( y = f(x) \), \( f(5) = -6 \)
So, \( f(g(1)) = -6 \)

6. Given the functions \( f(x) = 3x + 1 \) and \( g(x) = x^2 - 4 \), determine each value.

a) \( f(g(2)) \)

\[
\begin{align*}
g(2) & = 2^2 - 4 \\
& = 0 \\
f(g(2)) & = f(0) \\
& = 3(0) + 1 \\
& = 1 \\
So, f(g(2)) & = 1
\end{align*}
\]

b) \( g(f(2)) \)

\[
\begin{align*}
f(2) & = 3(2) + 1 \\
& = 7 \\
g(f(2)) & = g(7) \\
& = 7^2 - 4 \\
& = 45 \\
So, g(f(2)) & = 45
\end{align*}
\]

c) \( g(g(2)) \)

From part a, \( g(2) = 0 \)
\[
\begin{align*}
g(g(2)) & = g(0) \\
& = 0^2 - 4 \\
& = -4 \\
So, g(g(2)) & = -4
\end{align*}
\]

d) \( f(f(2)) \)

From part b, \( f(2) = 7 \)
\[
\begin{align*}
f(f(2)) & = f(7) \\
& = 3(7) + 1 \\
& = 22 \\
So, f(f(2)) & = 22
\end{align*}
\]
7. Given the graphs of \( y = f(x) \) and \( y = g(x) \), determine each value below.
   
   a) \( f(g(4)) \)
   
   From the graph of \( y = g(x) \), \( g(4) = 3 \)
   From the graph of \( y = f(x) \), \( f(3) = -2 \)
   So, \( f(g(4)) = -2 \)

   b) \( g(f(3)) \)
   
   From the graph of \( y = f(x) \), \( f(3) = 2 \)
   From the graph of \( y = g(x) \), \( g(-2) = -3 \)
   So, \( g(f(3)) = -3 \)

8. Given the functions \( f(x) = x^2 - 5x + 5 \) and \( g(x) = \frac{2x + 3}{x - 1} \), determine each value.
   
   a) \( f(g(-4)) \)
   
   \[
g(-4) = \frac{2(-4) + 3}{-4 - 1} = 1
   \]
   \[
f(1) = 1^2 - 5(1) + 5 = 1
   \]
   So, \( f(g(-4)) = 1 \)

   b) \( g(f(2)) \)
   
   \[
f(2) = 2^2 - 5(2) + 5 = -1
   \]
   \[
g(-1) = \frac{2(-1) + 3}{-1 - 1} = -0.5
   \]
   So, \( g(f(2)) = -0.5 \)

9. Given the functions \( f(x) = |4 - x| \), \( g(x) = (x - 4)^3 \), and \( h(x) = \sqrt{x} \), determine each value.
   
   a) \( f(g(1)) \)
   
   \[
g(1) = (1 - 4)^3 = 9
   \]
   \[
f(9) = |4 - 9| = 5
   \]
   So, \( f(g(1)) = 5 \)

   b) \( h(g(-2)) \)
   
   \[
g(-2) = (-2 - 4)^3 = 36
   \]
   \[
h(36) = \sqrt{36} = 6
   \]
   So, \( h(g(-2)) = 6 \)

   c) \( f(g(h(2))) \)
   
   \[
h(2) = \sqrt{2}
   \]
   \[
g(\sqrt{2}) = (\sqrt{2} - 4)^3 = 2 - 8\sqrt{2} + 16 = 18 - 8\sqrt{2}
   \]
   \[
f(18 - 8\sqrt{2}) = |4 - 18 + 8\sqrt{2}| = |14 + 8\sqrt{2}| = 14 - 8\sqrt{2}
   \]
   So, \( f(g(h(2))) = 14 - 8\sqrt{2} \)

   d) \( h(g(f(2))) \)
   
   \[
f(2) = |4 - 2| = 2
   \]
   \[
g(2) = (2 - 4)^3 = 4
   \]
   \[
h(4) = \sqrt{4} = 2
   \]
   So, \( h(g(f(2))) = 2 \)
10. Given \( f(x) = 4x - 3 \) and \( g(x) = -2x^2 + 3x \), determine an explicit equation for each composite function, then state its domain and range.

a) \( f(g(x)) \)
\[
\begin{align*}
f(g(x)) &= f(-2x^2 + 3x) \\
f(g(x)) &= 4(-2x^2 + 3x) - 3 \\
f(g(x)) &= -8x^2 + 12x - 3
\end{align*}
\]
This is a quadratic function; its domain is: \( x \in \mathbb{R} \); and its range is: \( y \leq 1.5 \)

b) \( g(f(x)) \)
\[
\begin{align*}
g(f(x)) &= g(4x - 3) \\
g(f(x)) &= -2(4x - 3)^2 + 3(4x - 3) \\
g(f(x)) &= -32x^2 + 48x - 18 + 12x - 9
\end{align*}
\]
This is a quadratic function; its domain is: \( x \in \mathbb{R} \); and its range is: \( y \leq 1.25 \)

c) \( g(g(x)) \)
\[
\begin{align*}
g(g(x)) &= g(-2x^2 + 3x) \\
g(g(x)) &= -2(-2x^2 + 3x)^2 + 3(-2x^2 + 3x) \\
&= -8x^4 + 24x^3 - 18x^2 - 6x^2 + 9x \\
&= -8x^4 + 24x^3 - 24x^2 + 9x
\end{align*}
\]
This is a polynomial function; its domain is: \( x \in \mathbb{R} \)
Use graphing technology to graph the function; its range is: \( y \leq 1.125 \)

d) \( f(f(x)) \)
\[
\begin{align*}
f(f(x)) &= f(4x - 3) \\
f(f(x)) &= 4(4x - 3) - 3 \\
f(f(x)) &= 16x - 15
\end{align*}
\]
This is a linear function; its domain is: \( x \in \mathbb{R} \); and its range is: \( y \in \mathbb{R} \)

11. Given \( f(x) = x^3 - 5 \) and \( g(x) = 3x + 1 \), determine an explicit equation for each composite function, then state its domain and range.

a) \( f(g(x)) \)
\[
\begin{align*}
f(g(x)) &= f(3x + 1) \\
f(g(x)) &= (3x + 1)^3 - 5
\end{align*}
\]
This is a cubic function; its domain is: \( x \in \mathbb{R} \); and its range is: \( y \in \mathbb{R} \)

b) \( g(f(x)) \)
\[
\begin{align*}
g(f(x)) &= g(x^3 - 5) \\
g(f(x)) &= 3(x^3 - 5) + 1 \\
g(f(x)) &= 3x^3 - 14
\end{align*}
\]
This is a cubic function; its domain is: \( x \in \mathbb{R} \); and its range is: \( y \in \mathbb{R} \)

c) \( f(f(x)) \)
\[
\begin{align*}
f(f(x)) &= f(x^3 - 5) \\
f(f(x)) &= (x^3 - 5)^3 - 5
\end{align*}
\]
This is a polynomial function with an odd degree; its domain is: \( x \in \mathbb{R} \); and its range is: \( y \in \mathbb{R} \)

d) \( g(g(x)) \)
\[
\begin{align*}
g(g(x)) &= g(3x + 1) \\
g(g(x)) &= 3(3x + 1) + 1 \\
g(g(x)) &= 9x + 4
\end{align*}
\]
This is a linear function; its domain is: \( x \in \mathbb{R} \); and its range is: \( y \in \mathbb{R} \)

No, two linear functions have the form \( f(x) = mx + b \) and \( g(x) = nx + c \). When I compose functions, I substitute one function for the variable in the other function, so for two linear functions, the composite function is also a linear function. For example, \( f(g(x)) = mnx + mc + b \), which simplifies to \( f(g(x)) = mnx + mc + b \).

13. Given the graphs of \( y = f(x) \) and \( y = g(x) \)

a) Determine \( f(g(2)) \) and \( g(f(2)) \).

From the graph:
- \( g(2) = 3 \) and \( f(2) = 0 \)
- \( f(3) = 2 \) and \( g(0) = 2 \)
- So, \( f(g(2)) = 2 \) and \( g(f(2)) = 2 \)

b) Determine \( f(g(1)) \) and \( g(f(1)) \).

From the graph:
- \( g(1) = 2.5 \) and \( f(1) = -2 \)
- \( f(2.5) = 1 \) and \( g(-2) = 1 \)
- So, \( f(g(1)) = 1 \) and \( g(f(1)) = 1 \)

c) How are the functions \( f(x) \) and \( g(x) \) related? Justify your answer.

From parts a and b, \( f(g(2)) = g(f(2)) = 2 \) and \( f(g(1)) = g(f(1)) = 1 \). The functions are inverses of each other. Their graphs are reflections of each other in the line \( y = x \).

14. Use composition of functions to determine whether the functions in each pair are inverse functions.

a) \( y = \frac{1}{3}x + 2 \) and \( y = 3x - 6 \)  

b) \( y = 2x - 3 \) and \( y = 2x + 3 \)

a) Write \( y = \frac{1}{3}x + 2 \) as \( f(x) = \frac{1}{3}x + 2 \), and write \( y = 3x - 6 \) as \( g(x) = 3x - 6 \).

Determine:
- \( f(g(x)) = \frac{1}{3}(3x - 6) + 2 = x \)

Determine:
- \( g(f(x)) = 3\left(\frac{1}{3}x + 2\right) - 6 = x \)

Since \( f(g(x)) \neq x \), the functions are not inverses.

b) Write \( y = 2x - 3 \) as \( f(x) = 2x - 3 \), and write \( y = 2x + 3 \) as \( g(x) = 2x + 3 \).

Determine:
- \( f(g(x)) = 2(2x + 3) - 3 = 4x + 3 \)

Since \( f(g(x)) \neq x \), the functions are not inverses.
15. Given the graphs of \( y = f(x) \) and \( y = g(x) \):

a) Determine the value of \( a \) for which 
\( f(g(a)) = -1 \).

Work backward. 
Determine the value of \( x \) for which 
\( f(x) = -1 \). 
From the graph, \( x = 1 \) 
Determine the value of \( x \) for which \( g(x) = 1 \). 
From the graph, \( x = 6 \) 
So, for \( f(g(a)) = -1, a = 6 \)

b) Determine the value of \( a \) for which \( g(f(a)) = 2 \).

Determine the value of \( x \) for which \( g(x) = 2 \). 
From the graph, \( x = 3 \) 
Determine the value of \( x \) for which \( f(x) = 3 \). 
From the graph, \( x = 5 \) 
So, for \( g(f(a)) = 2, a = 5 \)

16. Given the functions \( f(x) = x^2 - 2x \) and \( g(x) = 3x + 2 \), write an explicit expression for each value.

a) \( g(f(a)) \)

\[ g(f(x)) = 3(x^2 - 2x) + 2 \]
\[ g(f(x)) = 3x^2 - 6x + 2 \]
Substitute: \( x = a \)
\[ g(f(a)) = 3a^2 - 6a + 2 \]

b) \( f(g(a)) \)

\[ f(g(x)) = (3x + 2)^2 - 2(3x + 2) \]
\[ f(g(x)) = 9x^2 + 12x + 4 - 6x - 4 \]
\[ f(g(x)) = 9x^2 + 6x \]
Substitute: \( x = a \)
\[ f(g(a)) = 9a^2 + 6a \]

c) \( f(g(a - 1)) \)

From part b, \( f(g(x)) = 9x^2 + 6x \) 
Substitute: \( x = a - 1 \)
\[ f(g(a - 1)) = 9(a - 1)^2 + 6(a - 1) \]
\[ f(g(a - 1)) = 9a^2 - 18a + 9 + 6a - 6 \]
\[ f(g(a - 1)) = 9a^2 - 12a + 3 \]

d) \( f(g(1 - a)) \)

From part b, \( f(g(x)) = 9x^2 + 6x \) 
Substitute: \( x = 1 - a \)
\[ f(g(1 - a)) = 9(1 - a)^2 + 6(1 - a) \]
\[ f(g(1 - a)) = 9 - 18a + 9a^2 + 6 - 6a \]
\[ f(g(1 - a)) = 15 - 24a + 9a^2 \]
Write an explicit equation for \( g(h(x)) \).

\[
\begin{align*}
g(h(x)) &= 5\sqrt{x + 3} - 2 \\
g(h(x)) &= 5\sqrt{x + 3} - 2 \quad \text{Substitute: } x = a, g(h(a)) = 13 \\
13 &= 5\sqrt{a + 3} - 2 \\
5\sqrt{a + 3} &= 15 \\
\sqrt{a + 3} &= 3 \\
a + 3 &= 9 \\
a &= 6
\end{align*}
\]
Verify the solution.

b) Determine the values of \( a \) for which \( f(g(a)) = 5 \).

Write an explicit equation for \( f(g(x)) \).

\[
\begin{align*}
f(g(x)) &= (5x - 2)^2 - 2(5x - 2) + 2 \\
f(g(x)) &= 25x^2 - 20x + 4 - 10x + 4 + 2 \\
f(g(x)) &= 25x^2 - 30x + 10 \quad \text{Substitute: } x = a, f(g(a)) = 5 \\
5 &= 25a^2 - 30a + 10 \\
0 &= 25a^2 - 30a + 5 \quad \text{Factor.} \\
0 &= 5(5a^2 - 6a + 1) \\
0 &= 5(5a - 1)(a - 1) \\
a &= 0.2 \text{ or } a = 1
\end{align*}
\]

In part a, \( g(h(x)) \) is a radical function and its related equation has only one solution. In part b, \( f(g(x)) \) is a quadratic function and its related equation has two solutions.