Lesson 3.4 Exercises, pages 226–232

A

3. The graph of \( y = g(x) \) is the image of the graph of \( y = f(x) \) after a
single transformation. Identify the transformation.

a) A horizontal compression by a factor of \( \frac{1}{3} \), or a vertical
stretch by a factor of 3

b) A vertical compression by a factor of \( \frac{1}{2} \)
4. Describe how the graph of each function below is related to the graph of \( y = f(x) \).

a) \( y + 5 = -2f(x) \)

Compare \( y - k = af(b(x - h)) \) to \( y + 5 = -2f(x); \ k = -5, a = -2 \)

So, the graph of \( y = f(x) \) is vertically stretched by a factor of 2, reflected in the \( x \)-axis, then translated 5 units down.

b) \( y = f(3(x - 4)) \)

Compare \( y - k = af(b(x - h)) \) to \( y = f(3(x - 4)); b = 3, h = 4 \)

So, the graph of \( y = f(x) \) is horizontally compressed by a factor of \( \frac{1}{3} \), then translated 4 units right.

c) \( y = \frac{1}{2}f(x + 7) \)

Compare \( y - k = af(b(x - h)) \) to \( y = \frac{1}{2}f(x + 7); a = \frac{1}{2}, h = -7 \)

So, the graph of \( y = f(x) \) is vertically compressed by a factor of \( \frac{1}{2} \), then translated 7 units left.

d) \( y - 2 = f\left(\frac{1}{3}x\right) \)

Compare \( y - k = af(b(x - h)) \) to \( y - 2 = f\left(\frac{1}{3}x\right); k = 2, b = \frac{1}{3} \)

So, the graph of \( y = f(x) \) is horizontally stretched by a factor of 3, then translated 2 units up.

5. The graph of \( y = f(x) \) is transformed as described below. Write the equation of the image graph in terms of the function \( f \).

a) a horizontal compression by a factor of \( \frac{1}{4} \), a reflection in the \( y \)-axis, and a translation of 3 units left

The equation of the image graph has the form: \( y - k = af(b(x - h)) \)

Since \( b = -4 \) and \( h = -3 \), the equation is: \( y = f(-4(x + 3)) \)

b) a vertical compression by a factor of \( \frac{1}{2} \), a reflection in the \( y \)-axis, and a translation of 7 units up

The equation of the image graph has the form: \( y - k = af(b(x - h)) \)

Since \( a = \frac{1}{2}, b = -1 \), and \( k = 7 \), the equation is: \( y - 7 = \frac{1}{2}f(-x) \)

c) a horizontal stretch by a factor of 5, a vertical compression by a factor of \( \frac{1}{3} \), and a translation of 6 units left and 3 units up

In \( y - k = af(b(x - h)) \), substitute \( b = \frac{1}{5}, a = \frac{1}{3}, h = -6, \) and \( k = 3 \).

The equation is: \( y - 3 = \frac{1}{5}f\left(\frac{1}{3}(x + 6)\right) \)
Perform the vertical stretch by a factor of 3 first. Point \((x, y)\) on \(y = f(x)\) corresponds to point \((x, 3y)\) on the image graph \(y = 3f(x)\).

<table>
<thead>
<tr>
<th>Point on (y = f(x))</th>
<th>Point on (y = 3f(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-5, 3))</td>
<td>((-5, 9))</td>
</tr>
<tr>
<td>((-3, 1))</td>
<td>((-3, 3))</td>
</tr>
<tr>
<td>((1, 2))</td>
<td>((1, 6))</td>
</tr>
<tr>
<td>((3, -2))</td>
<td>((3, -6))</td>
</tr>
</tbody>
</table>

Plot the points, then join them in order with line segments to form the graph of \(y = 3f(x)\). Then translate this graph 4 units left and 2 units down to form the graph of \(y + 2 = 3f(x + 4)\).

7. Here is the graph of \(y = f(x)\). On the same grid, sketch the graph of each function below then state its domain and range.

a) \(y - 3 = \frac{1}{2}f(2(x + 1))\)

Compare: \(y - k = af(b(x - h))\) to \(y - 3 = \frac{1}{2}f(2(x + 1))\)

\(k = 3, a = \frac{1}{2}, b = 2, \) and \(h = -1\)

\((x, y)\) corresponds to \(\left(\frac{x}{2} - 1, -\frac{1}{2}y + 3\right)\)

<table>
<thead>
<tr>
<th>Point on (y = f(x))</th>
<th>Point on (y - 3 = \frac{1}{2}f(2(x + 1)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-4, 4))</td>
<td>((-3, 1))</td>
</tr>
<tr>
<td>((0, 0))</td>
<td>((-1, 3))</td>
</tr>
<tr>
<td>((4, 4))</td>
<td>((1, 1))</td>
</tr>
</tbody>
</table>

The domain is: \(x \in \mathbb{R}\)
The range is: \(y \leq 3\)
b) \( y + 1 = 3f(-(x - 4)) \)

Compare: \( y - k = af(b(x - h)) \) to \( y + 1 = 3f(-(x - 4)) \)

\( k = -1, a = 3, b = -1, \) and \( h = 4 \)

\((x, y)\) corresponds to \((-x + 4, 3y - 1)\)

<table>
<thead>
<tr>
<th>Point on ( y = f(x) )</th>
<th>Point on ( y + 1 = 3f(-(x - 4)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-4, 4))</td>
<td>((8, 11))</td>
</tr>
<tr>
<td>((0, 0))</td>
<td>((4, -1))</td>
</tr>
<tr>
<td>((4, 4))</td>
<td>((0, 11))</td>
</tr>
</tbody>
</table>

The domain is: \( x \in \mathbb{R} \)
The range is: \( y \geq -1 \)

8. On each grid, graph \( y = \sqrt{x} \), apply transformations to sketch the given function, then state its domain and range.

a) \( y = -\sqrt{x} + 2 \)

\( y = -\sqrt{x} - (-2) \)
\( a = -1 \) and \( h = -2 \)

\((x, y)\) corresponds to \((-2, -y)\)

<table>
<thead>
<tr>
<th>Point on ( y = \sqrt{x} )</th>
<th>Point on ( y = -\sqrt{x} + 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>((-2, 0))</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>((-1, -1))</td>
</tr>
<tr>
<td>((9, 3))</td>
<td>((7, -3))</td>
</tr>
</tbody>
</table>

Domain is: \( x \geq -2 \)
Range is: \( y \leq 0 \)

b) \( y + 5 = -2\sqrt{3(x - 1)} \)

\( k = -5, a = -2, b = 3, \) and \( h = 1 \)

\((x, y)\) corresponds to \( \left( \frac{x}{3} + 1, -2y - 5 \right) \)

<table>
<thead>
<tr>
<th>Point on ( y = \sqrt{x} )</th>
<th>Point on ( y + 5 = -2\sqrt{3(x - 1)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0))</td>
<td>((1, -5))</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>(\left( \frac{4}{3}, -7 \right))</td>
</tr>
<tr>
<td>((9, 3))</td>
<td>((4, -11))</td>
</tr>
</tbody>
</table>

Domain is: \( x \geq 1 \)
Range is: \( y \leq -5 \)
9. The graph of \( y = g(x) \) is the image of the graph of \( y = f(x) \) after a combination of transformations. Corresponding points are labelled. Write an equation of each image graph in terms of the function \( f \).

\[ \text{a) A}(0, 9) \text{ A}(1, 0) \]

Write the equation for the image graph in the form \( y - k = af(b(x - h)) \).

Use the points \( A(0, 9) \) and \( B(3, 0) \) on the graph of \( y = f(x) \).

Horizontal distance between \( A \) and \( B \) is: \( 3 \)

Vertical distance between \( A \) and \( B \) is: \( 9 \)

Use corresponding points \( A'(1, -5) \) and \( B'(2, 4) \) on the graph of \( y = g(x) \).

Horizontal distance between \( A' \) and \( B' \) is: \( 1 \)

Vertical distance between \( A' \) and \( B' \) is: \( 9 \)

The horizontal distance is one-third of the original distance, so the graph of \( y = f(x) \) is compressed horizontally by a factor of \( \frac{1}{3} \): \( b = 3 \).

The vertical distance does not change, so the graph of \( y = f(x) \) is not compressed or stretched vertically. From the graph, there is a reflection in the \( x \)-axis, so \( a = -1 \). To determine the coordinates of \( B(3, 0) \) after this compression and reflection, substitute: \( x = 3, y = 0, a = -1, \) and \( b = 3 \) in \( \left( \frac{x}{b}, ay \right) \) to get \( \left( \frac{3}{3}, 0 \right) \), or \( (1, 0) \). Determine the translation that would move \((1, 0)\) to \( B'(2, 4) \).

A translation of 1 unit right and 4 units up is required, so \( h = 1 \) and \( k = 4 \).

An equation for the image graph is: \( y - 4 = -f(3(x - 1)) \)

\[ \text{b) A}(2, 8) \text{ A}(3, 2) \]

Write the equation for the image graph in the form \( y - k = af(b(x - h)) \).

Use the points \( A(2, 8) \) and \( B(0, 0) \) on the graph of \( y = f(x) \).

Horizontal distance between \( A \) and \( B \) is: \( 2 \)

Vertical distance between \( A \) and \( B \) is: \( 8 \)

Use corresponding points \( A'(-5, 6) \) and \( B'(-3, 2) \) on the graph of \( y = g(x) \).

Horizontal distance between \( A' \) and \( B' \) is: \( 2 \)

Vertical distance between \( A' \) and \( B' \) is: \( 4 \)

The horizontal distance does not change, so the graph of \( y = f(x) \) is not compressed or stretched horizontally. From the graph, there is a reflection in the \( y \)-axis, so \( b = -1 \).

The vertical distance is halved, so the graph of \( y = f(x) \) is compressed vertically by a factor of \( \frac{1}{2} \): \( a = \frac{1}{2} \). To determine the coordinates of \( A(2, 8) \) after this compression and reflection, substitute: \( x = 2, y = 8, b = -1, \) and \( a = \frac{1}{2} \) in \( \left( \frac{x}{b}, ay \right) \) to get \( \left( \frac{2}{-1}, \frac{1}{2}(8) \right) \), or \((-2, 4)\).

Determine the translation that would move \((-2, 4)\) to \( A'(-5, 6) \).

A translation of 3 units left and 2 units up is required, so \( h = -3 \) and \( k = 2 \). An equation for the image graph is: \( y - 2 = \frac{1}{2}(-f(x + 3)) \)
10. For each pair of functions below, describe the graph of the second function as a transformation image of the graph of the first function.

a) \( y = |x| \quad y + 6 = -2|3(x - 4)| \)

Let \( f(x) = |x| \), then compare \( y + 6 = -2|3(x - 4)| \) to \( y - k = af(b(x - h)); k = -6, a = -2, b = 3, \) and \( h = 4 \).

The graph of \( y + 6 = -2|3(x - 4)| \) is the image of the graph of \( y = |x| \) after a vertical stretch by a factor of 2, a horizontal compression by a factor of \( \frac{1}{3} \), a reflection in the \( x \)-axis, followed by a translation of 4 units right and 6 units down.

b) \( y = \frac{1}{x} \quad y - 3 = 2\left(\frac{5}{x + 1}\right) \)

Let \( f(x) = \frac{1}{x} \), then compare \( y - 3 = 2\left(\frac{5}{x + 1}\right) \) to \( y - k = af(b(x - h)); k = 3, a = 2, b = 5, \) and \( h = -1 \).

The graph of \( y - 3 = 2\left(\frac{5}{x + 1}\right) \) is the image of the graph of \( y = \frac{1}{x} \) after a vertical stretch by a factor of 2, a horizontal compression by a factor of \( \frac{1}{5} \), followed by a translation of 1 unit left and 3 units up.

c) \( y = x^4 \quad y + 1 = \frac{1}{4}[-2(x + 3)]^4 \)

Let \( f(x) = x^4 \), then compare \( y + 1 = \frac{1}{4}[-2(x + 3)]^4 \) to \( y - k = af(b(x - h)); k = -1, a = \frac{1}{4}, b = -2, \) and \( h = -3 \).

The graph of \( y + 1 = \frac{1}{4}[-2(x + 3)]^4 \) is the image of the graph of \( y = x^4 \) after a vertical compression by a factor of \( \frac{1}{4} \), a horizontal compression by a factor of \( \frac{1}{2} \), a reflection in the \( y \)-axis, followed by a translation of 3 units left and 1 unit down.

11. A transformation image of the graph of \( y = f(x) \) is represented by the equation \( y - 1 = -2f\left(\frac{x + 5}{3}\right) \). The point \((7, 5)\) lies on the image graph. What are the coordinates of the corresponding point on the graph of \( y = f(x) \)?

Compare \( y - 1 = -2f\left(\frac{x + 5}{3}\right) \) to \( y - k = af(b(x - h)); k = -1, a = -2, b = \frac{1}{3}, \) and \( h = -5 \).

A point \((x, y)\) on \( y = f(x) \) corresponds to the point \((3x - 5, -2y + 1)\) on \( y - 1 = -2f\left(\frac{x + 5}{3}\right) \). The image of a point \((x, y)\) is \((7, 5)\).

So, \(3x - 5 = 7\), or \(x = 4\); and \(-2y + 1 = 5\), or \(y = -2\)

So, the corresponding point on \( y = f(x) \) is \((4, -2)\).
12. This graph is the image of the graph of 
\( y = |x| \) after a combination of 
transformations. Write an equation of the image.

Write the equation for the image graph in 
the form \( y - k = a\left|b(x - h)\right|\).
Sketch the graph of \( y = |x|\).
Use the points \( A(0, 0) \) and \( B(1, 1) \) on the 
graph of \( y = |x|\).
Horizontal distance between \( A \) and \( B \) is: 1
Vertical distance between \( A \) and \( B \) is: 1
Use corresponding points \( A'(−3, 4) \) and \( B'(−1, 3) \) on the image graph.
Horizontal distance between \( A' \) and \( B' \) is: 2
Vertical distance between \( A' \) and \( B' \) is: 1
The horizontal distance is doubled, so the graph of \( y = |x| \) is stretched 
horizontally by a factor of 2 and \( b = \frac{1}{2} \).
The vertical distance does not change, so the graph of \( y = |x| \) is not 
compressed or stretched vertically. From the graph, there is a reflection in 
the \( x \)-axis, so \( a = −1 \).
To determine the coordinates of \( B(1, 1) \) after this stretch and reflection, 
substitute: \( x = 1, y = 1, b = \frac{1}{2}, \) and \( a = −1 \) in \( \left(\frac{x}{b}, ay\right) \) to get \( (2, −1) \).
Determine the translation that would move \( (2, −1) \) to \( B'(-1, 3) \). 
A translation of 3 units left and 4 units up is required, so \( h = −3 \) and 
\( k = 4 \).
An equation for the image graph is: \( y − 4 = −\frac{1}{2}(x + 3) \), or 
\( y − 4 = −\frac{1}{2}|x + 3| \)
Use mental math to check this equation, by verifying that the point \( (1, 2) \) 
lies on the graph.