Lesson 1.4 Exercises, pages 46–54

A

3. Which functions are polynomial functions? Justify your choices.
   a) \( f(x) = 2\sqrt{x} - x^2 \)
      Not a polynomial function: \( \sqrt{x} = x^{\frac{1}{2}} \) and \( \frac{1}{2} \) is not a whole number.

   b) \( g(x) = 6x^3 - x^2 + 3x - 7 \)
      Polynomial function: the coefficients of the variables are real numbers and all exponents are whole numbers.

   c) \( h(x) = 7x^2 + 2x^3 - x - \frac{1}{2} \)
      Polynomial function: the coefficients of the variables are real numbers and all exponents are whole numbers.

   d) \( k(x) = 3^x + 5 \)
      Not a polynomial function: the variable \( x \) is an exponent.

   e) \( p(x) = 5x^2 - 7x + \frac{2}{x} \)
      Not a polynomial function: \( \frac{2}{x} = 2x^{-1} \) and the exponent is not a whole number.
4. Which graphs are graphs of polynomial functions? Justify your answers.

a) No, graph has sharp corners.

b) Yes, graph is smooth and continuous and has a possible shape for an odd-degree polynomial function.

c) No, graph is not continuous.

d) Yes, graph is smooth and continuous and has a possible shape for an odd-degree polynomial function.

5. Complete the table below. The first row has been done for you.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Degree</th>
<th>Odd or Even Degree</th>
<th>Type</th>
<th>Leading coefficient</th>
<th>y-intercept of its graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 3x^2 - 2x + 1 )</td>
<td>2</td>
<td>even</td>
<td>quadratic</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>a) ( g(x) = 5x + x^5 - 2x^3 )</td>
<td>5</td>
<td>odd</td>
<td>quintic</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>b) ( h(x) = 2x^3 - 3x^4 - 7 )</td>
<td>3</td>
<td>odd</td>
<td>cubic</td>
<td>-3</td>
<td>-7</td>
</tr>
<tr>
<td>c) ( k(x) = 5 - x^4 - 3x )</td>
<td>4</td>
<td>even</td>
<td>quartic</td>
<td>-1</td>
<td>5</td>
</tr>
</tbody>
</table>
6. Use a table of values to sketch the graph of each polynomial function.

a) \( f(x) = x^3 - 7x + 6 \)

The equation represents an odd-degree polynomial function. The leading coefficient is positive, so as \( x \to -\infty \), the graph falls and as \( x \to \infty \), the graph rises. The constant term is 6, so the \( y \)-intercept is 6.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>12</td>
</tr>
<tr>
<td>-1</td>
<td>12</td>
</tr>
<tr>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>

b) \( g(x) = -x^4 + 5x^2 - 4 \)

The equation represents an even-degree polynomial function. The leading coefficient is negative, so the graph opens down. The constant term is -4, so the \( y \)-intercept is -4.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-40</td>
</tr>
<tr>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-40</td>
</tr>
</tbody>
</table>
7. Use intercepts to sketch the graph of each polynomial function.

a) \( f(x) = 2x^3 + 3x^2 - 2x \)

Factor.
\[ f(x) = x(2x^2 + 3x - 2) \]
\[ f(x) = x(x + 2)(2x - 1) \]
Determine the zeros of \( f(x) \). Let \( f(x) = 0 \).
\[ 0 = x(x + 2)(2x - 1) \]
The zeros are: 0, \(-2, \frac{1}{2}\).
So, the \( x \)-intercepts of the graph are: 0, \(-2, \frac{1}{2}\).
The equation has degree 3, so it is an odd-degree polynomial function.
The leading coefficient is positive, so as \( x \rightarrow -\infty \), the graph falls and as \( x \rightarrow \infty \), the graph rises.
The constant term is 0, so the \( y \)-intercept is 0.

b) \( h(x) = 2x^4 + 7x^3 + 4x^2 - 7x - 6 \)

Factor the polynomial. Use the factor theorem.
The factors of the constant term, \(-6\), are: 1, \(-1, 2, -2, 3, -3, 6, -6\).
Use mental math to substitute \( x = 1 \), then \( x = -1 \) in \( h(x) \) to determine that both \( x - 1 \) and \( x + 1 \) are factors.
Divide by \( x - 1 \).

\[
\begin{array}{c|cccc}
1 & 2 & 7 & 4 & -7 & -6 \\
\hline
 & 2 & 9 & 13 & 6 & 0 \\
\end{array}
\]
So, \( 2x^4 + 7x^3 + 4x^2 - 7x - 6 = (x - 1)(2x^3 + 9x^2 + 13x + 6) \)
Divide \( 2x^3 + 9x^2 + 13x + 6 \) by \( x + 1 \).

\[
\begin{array}{c|ccc}
-1 & 2 & 9 & 13 & 6 \\
\hline
 & 2 & 7 & 6 & 0 \\
\end{array}
\]
So, \( 2x^4 + 7x^3 + 4x^2 - 7x - 6 = (x - 1)(x + 1)(2x^2 + 7x + 6) \)
Factor the trinomial: \( 2x^2 + 7x + 6 = (2x + 3)(x + 2) \)
So, \( 2x^4 + 7x^3 + 4x^2 - 7x - 6 = (x - 1)(x + 1)(2x + 3)(x + 2) \)
Determine the zeros of \( h(x) \). Let \( h(x) = 0 \).
\[ 0 = (x - 1)(x + 1)(2x + 3)(x + 2) \]
The zeros are: 1, \(-1, -1.5, -2\).
So, the \( x \)-intercepts of the graph are: 1, \(-1, -1.5, -2\).
The equation has degree 4, so it is an even-degree polynomial function.
The leading coefficient is positive, so the graph opens up. The constant term is \(-6\), so the \( y \)-intercept is \(-6\).
8. Identify the graph that corresponds to each function. Justify your choices.

a) \( f(x) = -x^3 + 3x^2 + x - 3 \)
   - Odd degree, negative leading coefficient: graph B

b) \( g(x) = x^4 - 3x^2 - 3 \)
   - Even degree, positive leading coefficient: graph D

c) \( h(x) = x^3 + 3x^2 - 3 \)
   - Odd degree, positive leading coefficient: graph A

d) \( k(x) = -x^2 + 4x - 3 \)
   - Even degree, negative leading coefficient: graph C

i) Graph A

ii) Graph B

iii) Graph C

iv) Graph D

9. Determine the zeros of each polynomial function. State the multiplicity of each zero. How does the graph of each function behave at the related x-intercepts? Use graphing technology to check.

a) \( f(x) = (x + 3)^3 \)
   - 0 = (x + 3)^3
   - Root of the equation: \( x = -3 \)
   - Zero of the function: -3
   - The zero has multiplicity 3.
   - So, the graph crosses the x-axis at \( x = -3 \).

b) \( g(x) = (x - 2)^2(x + 3)^2 \)
   - 0 = (x - 2)^2(x + 3)^2
   - Roots of the equation: \( x = 2 \) and \( x = -3 \)
   - Zeros of the function: 2 and -3
   - The zero 2 has multiplicity 2.
   - The zero -3 has multiplicity 2.
   - So, the graph just touches the x-axis at \( x = 2 \) and at \( x = -3 \).
10. Sketch the graph of this polynomial function.

\[ h(x) = (x + 1)^2(x - 1)(x + 2) \]

To determine the roots, let \( h(x) = 0 \).

- Roots of the equation: \( x = 1 \) and \( x = -0.5 \)
- Zeros of the function: \( 1 \) and \( -0.5 \)
- The zero \( 1 \) has multiplicity 4.
- The zero \( -0.5 \) has multiplicity 1.
- So, the graph just touches the \( x \)-axis at \( x = 1 \) and crosses the \( x \)-axis at \( x = -0.5 \).

\[ 0 = (x - 1)^2(2x + 1) \]

\[ 0 = (x - 4)^3(x + 1)^2 \]

11. a) Write an equation in standard form for each polynomial function described below.

i) a cubic function with zeros \( 3, -3, \) and \( 0 \)

- Sample response:
  - The zeros of the function are the roots of its equation.
  - \( y = x(x - 3)(x + 3) \)
  - \( y = x(x^2 - 9) \)
  - \( y = x^3 - 9x \)

ii) a quartic function with zeros \( -2, 1 \) of multiplicity 1, and a zero \( 2 \) of multiplicity 2

- Sample response:
  - The zeros of the function are the roots of its equation.
  - \( y = (x + 2)(x - 1)(x - 2)^2 \)
  - \( y = (x^2 + x - 2)(x^2 - 4x + 4) \)
  - \( y = x^4 - 4x^2 + 4x^2 + x^2 - 4x + 4x - 2x^2 + 8x - 8 \)
  - \( y = x^4 - 3x^2 - 2x^2 + 12x - 8 \)
b) Is there more than one possible equation for each function in part a? Explain.

Yes, if I multiply the polynomial by a constant factor, I don’t change the zeros but I do change the equation.

12. Sketch a possible graph of each polynomial function.

   a) cubic function; leading coefficient is positive; zero of 4 has multiplicity 3

   The zero has multiplicity 3, so the graph crosses the x-axis at $x = 4$. Since the function is cubic, there are no more zeros. The leading coefficient is positive so as $x \to -\infty$, the graph falls and as $x \to \infty$, the graph rises.

   b) quintic function; leading coefficient is positive; zero of 3 has multiplicity 2; zero of −2 has multiplicity 2; zero of −4 has multiplicity 1

   Each of the zeros 3 and −2 has multiplicity 2, so the graph just touches the x-axis at $x = 3$ and $x = −2$. The zero −4 has multiplicity 1, so the graph crosses the x-axis at $x = −4$. Since the function is quintic, there are no more zeros. The leading coefficient is positive, so as $x \to -\infty$, the graph falls and as $x \to \infty$, the graph rises.

   c) quartic function; leading coefficient is negative; zero of −4 has multiplicity 3; zero of 3 has multiplicity 1

   The zero −4 has multiplicity 3, so the graph crosses the x-axis at $x = −4$. The zero 3 has multiplicity 1, so the graph crosses the x-axis at $x = 3$. Since the function is quartic, there are no more zeros. The leading coefficient is negative, so the graph opens down.

13. A cubic function has zeros 2, 3, and −1. The y-intercept of its graph is −18. Sketch the graph, then determine an equation of the function.

   The zeros of the function are the roots of its equation.
   Let $k$ represent the leading coefficient.
   $y = k(x − 2)(x − 3)(x + 1)$
   The constant term in the equation is −18.
   So, $k(−2)(−3)(1) = −18$
   $k = −3$
   So, an equation is:
   $y = −3(x − 2)(x − 3)(x + 1)$
   $y = −3(x^2 − 5x + 6)(x + 1)$
   $y = −3x^3 + 12x^2 − 3x − 18$
14. Investigate pairs of graphs of even-degree polynomial functions of the form shown below for different values of the variables $a, b, c,$ and $d \in \mathbb{Z}$. Describe one graph as a transformation image of the other graph. What conclusions can you make?

$h(x) = (x + a)(x + b)(x + c)(x + d)$ and $k(x) = (x - a)(x - b)(x - c)(x - d)$

Sample response:

The graph of $k(x) = (x - 1)(x - 2)(x - 3)(x - 4)$ is the image of the graph of $h(x) = (x + 1)(x + 2)(x + 3)(x + 4)$ after a reflection in the $y$-axis. In general, the graph of $k(x)$ is the image of the graph of $h(x)$ after a reflection in the $y$-axis.

15. Investigate pairs of graphs of odd-degree polynomial functions of the form shown below for different values of the variables $a, b, c, d,$ and $e \in \mathbb{Z}$. Describe one graph as a transformation image of the other graph. What conclusions can you make?

$h(x) = (x + a)(x + b)(x + c)(x + d)(x + e)$ and $k(x) = (x - a)(x - b)(x - c)(x - d)(x - e)$

Sample response:

The graph of $k(x) = (x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$ is the image of the graph of $h(x) = (x + 1)(x + 2)(x + 3)(x + 4)(x + 5)$ after a rotation of $180^\circ$ about the origin. In general, the graph of $k(x)$ is the image of the graph of $h(x)$ after a rotation of $180^\circ$ about the origin.
16. Each of the functions \( f(x) = x^3 - 27x + 54 \) and 
\( g(x) = x^3 - 27x - 54 \) has one zero of multiplicity 2 and one 
different zero. Use only this information to determine the values of 
\( b \) for which the function \( h(x) = x^3 - 27x + b \) has each number of 
zeros. Explain your strategy.

a) 3 different zeros

I sketched the graphs.

\[ f(x) = x^3 - 27x + 54 \]
\[ g(x) = x^3 - 27x - 54 \]

Each of the graphs of \( f(x) \) and \( g(x) \) just touches the \( x \)-axis at the point 
that corresponds to the zero of multiplicity 2. The graph of \( f(x) \) has 
\( y \)-intercept 54 and the graph of \( g(x) \) has \( y \)-intercept -54. So, for the 
graph of \( h(x) \) to have 3 different zeros, the graph of \( h(x) \) must lie 
between the graphs of \( f(x) \) and \( g(x) \). So, \(-54 < b < 54\)

b) 1 zero of multiplicity 1 and no other zeros

For the graph of \( h(x) \) to have 1 zero of multiplicity 1 and no other zeros, 
the graph of \( f(x) \) must be translated up or the graph of \( g(x) \) must be 
translated down so that the local minimum point lies above the \( x \)-axis or 
the local maximum point lies below the \( x \)-axis. So, \( b > 54 \) or \( b < -54 \)