Lesson 1.2 Exercises, pages 20–25

A

3. Write each binomial in the form $x - a$. What is the value of $a$?

   a) $x + 4$  
   $x + 4 = x - (-4)$  
   $a = -4$

   b) $x - 1$  
   $x - 1$ is in the form $x - a$.  
   $a = 1$

   c) $11 + x$  
   $11 + x = x - (-11)$  
   $a = -11$

   d) $-7 + x$  
   $-7 + x = x - 7$  
   $a = 7$

4. a) Determine the remainder when $x^3 - 4x^2 - 7x + 10$ is divided by each binomial.

   i) $x - 1$  
   Let $P(x) = x^3 - 4x^2 - 7x + 10$
   $P(1) = 1^3 - 4(1)^2 - 7(1) + 10 = 0$
   The remainder is 0.

   ii) $x + 3$
   $P(-3) = (-3)^3 - 4(-3)^2 - 7(-3) + 10 = -32$
   The remainder is $-32$.

   iii) $x + 2$
   $P(-2) = (-2)^3 - 4(-2)^2 - 7(-2) + 10 = 0$
   The remainder is 0.

   iv) $x - 2$
   $P(2) = 2^3 - 4(2)^2 - 7(2) + 10 = -12$
   The remainder is $-12$. 
b) Which binomials in part a are factors of \(x^3 - 4x^2 - 7x + 10\)?

How do you know?

\(x - 1\) and \(x + 2\) are factors of \(x^3 - 4x^2 - 7x + 10\) because the value of the polynomial when \(x = 1\) and when \(x = -2\) is 0.

5. Which values of \(a, a \in \mathbb{Z}\), should be chosen to test for binomial factors of the form \(x - a\) of the polynomial \(x^4 + 3x^3 - 8x^2 - 12x + 16\)?

How did you choose the values?

I chose values of \(a\) that are factors of the constant term in the polynomial, 16. Factors of 16 are: 1, -1, 2, -2, 4, -4, 8, -8, 16, -16

6. a) Determine the remainder when each polynomial is divided by \(x - 2\).

i) \(x^3 - 7x + 11\)

Let \(P(x) = x^3 - 7x + 11\)

\[P(2) = (2)^3 - 7(2) + 11 = 4 - 14 + 11 = 1\]

The remainder is 1.

ii) \(2x^3 - 3x^2 - 6x + 8\)

Let \(P(x) = 2x^3 - 3x^2 - 6x + 8\)

\[P(2) = 2(2)^3 - 3(2)^2 - 6(2) + 8 = 16 - 12 - 12 + 8 = 0\]

The remainder is 0.

iii) \(3x^3 - 2x^2 - 10x + 6\)

Let \(P(x) = 3x^3 - 2x^2 - 10x + 6\)

\[P(2) = 3(2)^3 - 2(2)^2 - 10(2) + 6 = 24 - 8 - 20 + 6 = 2\]

The remainder is 2.

iv) \(x^4 - 2x^3 + 3x^2 - 8\)

Let \(P(x) = x^4 - 2x^3 + 3x^2 - 8\)

\[P(2) = (2)^4 - 2(2)^3 + 3(2)^2 - 8 = 16 - 16 + 12 - 8 = 4\]

The remainder is 4.

b) Explain the relationship between the remainder when a polynomial \(P(x)\) is divided by \(x - a, a \in \mathbb{Z}\), and \(P(a)\).

When a polynomial \(P(x)\) is divided by \(x - a\), the remainder is \(P(a)\).

This result comes from the division statement: \(P(x) = (x - a)Q(x) + R\)

When \(x = a\), \(x - a = 0\), so \((x - a)Q(x) = 0\)

Then, \(P(a) = R\)
7. Determine the remainder.
   a) \((2x^3 - x^2 + 3x - 2) \div (x - 3)\)
   b) \((3x^3 - 2x^2 - 4x + 6) \div (x - 2)\)

   Let \(P(x) = 2x^3 - x^2 + 3x - 2\)
   \(P(3) = 2(3)^3 - (3)^2 + 3(3) - 2 = 54 - 9 + 9 - 2 = 52\)
   The remainder is 52.

   Let \(P(x) = 3x^3 - 2x^2 - 4x + 6\)
   \(P(2) = 3(2)^3 - 2(2)^2 - 4(2) + 6 = 24 - 8 - 8 + 6 = 14\)
   The remainder is 14.

8. When \(2x^3 + kx^2 - 3x + 2\) is divided by \(x - 2\), the remainder is 4. Determine the value of \(k\).

   Let \(P(x) = 2x^3 + kx^2 - 3x + 2\)
   \(P(2) = 2(2)^3 + k(2)^2 - 3(2) + 2 = 16 + 4k - 6 + 2 = 12 + 4k\)
   The remainder is 4.

   So, \(12 + 4k = 4\) Solve for \(k\).
   \(4k = -8\)
   \(k = -2\)
   The value of \(k\) is -2.

9. Determine one binomial factor of each polynomial.
   a) \(x^4 + 6x^3 + 5x^2 - 24x - 36\)

   Sample response:
   Let \(P(x) = x^4 + 6x^3 + 5x^2 - 24x - 36\)
   The factors of -36 are: \(1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 9, -9, 12, -12, 18, -18, 36, -36\)
   Use mental math to substitute \(x = 1\), then \(x = -1\) to determine that neither \(x - 1\) nor \(x + 1\) is a factor.
   Try \(x = 2:\) \(P(2) = (2)^4 + 6(2)^3 + 5(2)^2 - 24(2) - 36 = 0\)
   So, \(x - 2\) is a factor of \(x^4 + 6x^3 + 5x^2 - 24x - 36\).

   b) \(x^4 + 3x^3 - 5x^2 + 15x + 12\)

   Sample response:
   Let \(P(x) = x^4 + 3x^3 - 5x^2 + 15x + 12\)
   The factors of 12 are: \(1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 12, -12\)
   Use mental math to substitute \(x = 1:\)
   \(P(1) = 0\)
   So, \(x - 1\) is a factor of \(x^4 + 3x^3 - 5x^2 + 15x + 12\).
10. a) Show that $x + 5$ is a factor of $x^3 + 4x^2 - 11x - 30$.

Let $P(x) = x^3 + 4x^2 - 11x - 30$

$P(-5) = (-5)^3 + 4(-5)^2 - 11(-5) - 30$

$= 0$

The remainder is 0, so $x + 5$ is a factor of $x^3 + 4x^2 - 11x - 30$.

b) Determine the other binomial factors of the polynomial.

Verify that the factors are correct.

Divide by $x + 5$ to determine the other factor.

\[
\begin{array}{c|cccc}
-5 & 1 & 4 & -11 & -30 \\
\hline
 & -5 & 5 & 30 \\
1 & 1 & -1 & -6 & 0 \\
\end{array}
\]

So, $x^3 + 4x^2 - 11x - 30 = (x + 5)(x^2 - x - 6)$

Factor the trinomial.

$x^2 - x - 6 = (x + 2)(x - 3)$

So, $x^3 + 4x^2 - 11x - 30 = (x + 5)(x - 3)(x + 2)$

To verify, expand:

$(x + 2)(x - 3)(x + 5) = (x^2 - x - 6)(x + 5)$

$= x^3 + 5x^2 - x^2 - 5x - 6x - 30$

$= x^3 + 4x^2 - 11x - 30$

Since this is the original polynomial, the factors are correct.

11. Fully factor each polynomial.

a) $x^3 + 6x^2 + 3x - 10$

Let $P(x) = x^3 + 6x^2 + 3x - 10$

The factors of $-10$ are: 1, -1, 2, -2, 5, -5, 10, -10

Use mental math to substitute $x = 1$:

$P(1) = 0$

So, $x - 1$ is a factor.

Divide to determine the other factor.

\[
\begin{array}{c|cccc}
1 & 1 & 6 & 3 & -10 \\
\hline
 & 1 & 7 & 10 \\
1 & 1 & 7 & 10 & 0 \\
\end{array}
\]

So, $x^3 + 6x^2 + 3x - 10 = (x - 1)(x^2 + 7x + 10)$

Factor the trinomial: $x^2 + 7x + 10 = (x + 2)(x + 5)$

So, $x^3 + 6x^2 + 3x - 10 = (x - 1)(x + 2)(x + 5)$
b) \( x^4 - 5x^2 + 4 \)

Let \( P(x) = x^4 - 5x^2 + 4 \)

The factors of 4 are: 1, -1, 2, -2, 4, -4

Use mental math to substitute \( x = 1 \):
\[ P(1) = 0; \text{ so, } x - 1 \text{ is a factor.} \]

Use mental math to substitute \( x = -1 \):
\[ P(-1) = 0; \text{ so, } x + 1 \text{ is a factor.} \]

Try \( x = 2: P(2) = (2)^4 - 5(2)^2 + 4 \)
\[ = 0 \]
So, \( x - 2 \) is a factor.

Try \( x = -2: P(-2) = (-2)^4 - 5(-2)^2 + 4 \)
\[ = 0 \]
So, \( x + 2 \) is a factor.

Since the original polynomial has degree 4, it can have at most 4 binomial factors.

So, \( x^4 - 5x^2 + 4 = (x - 1)(x + 1)(x - 2)(x + 2) \)

12. a) What value of \( b \) will ensure \( x + 3 \) is a factor of \( bx^3 - 2x^2 + x - 6 \)?

Let \( P(x) = bx^3 - 2x^2 + x - 6 \)

If \( x + 3 \) is a factor, \( P(-3) = 0 \)
\[ P(-3) = b(-3)^3 - 2(-3)^2 + (-3) - 6 \]
\[ = -27b - 27 \]
Let \( P(-3) = 0 \)
\[ -27b - 27 = 0 \]
\[ b = -1 \]
So, the value of \( b \) is -1.

b) What value of \( d \) will ensure \( x + 2 \) is a factor of \( 3x^3 - dx^4 + 4x^3 - 2dx^2 + x + 10 \)?

Let \( P(x) = 3x^3 - dx^4 + 4x^3 - 2dx^2 + x + 10 \)

If \( x + 2 \) is a factor, \( P(-2) = 0 \)
\[ P(-2) = 3(-2)^3 - d(-2)^4 + 4(-2)^3 - 2d(-2)^2 + (-2) + 10 \]
\[ = -120 - 24d \]
Let \( P(-2) = 0 \)
\[ -120 - 24d = 0 \]
\[ d = \frac{120}{24}; \text{ or } -5 \]
So, the value of \( d \) is -5.

13. Determine whether \( x + b \) is a factor of \((x + b)^3 + (x + p)^3 + (b - p)^3\), \( b, p \in \mathbb{R} \).

Let \( P(x) = (x + b)^3 + (x + p)^3 + (b - p)^3 \)

If \( x + b \) is a factor, \( P(-b) = 0 \)
\[ P(-b) = (-b + b)^3 + (-b + p)^3 + (b - p)^3 \]
\[ = 0 + (-b + p)^3 + (b - p)^3 \]
\[ = 0 - (b - p)^3 + (b - p)^3 \]
\[ = 0 \]

Since the remainder is 0, \( (x + b) \) is a factor of \((x + b)^3 + (x + p)^3 + (b - p)^3\), \( b, p \in \mathbb{R} \).
14. When $mx^3 - 2x^2 + nx - 4$ is divided by $x + 2$, the remainder is 4. When $mx^3 - 2x^2 + nx - 4$ is divided by $x - 1$, the remainder is $-11$. Determine the values of $m$ and $n$.

Let $P(x) = mx^3 - 2x^2 + nx - 4$

$P(-2) = 4$

$P(-2) = m(-2)^3 - 2(-2)^2 + n(-2) - 4$

$4 = -8m - 2n - 12$

$0 = -8m - 2n - 16 \ \odot$

$P(1) = -11$

$P(1) = m(1)^3 - 2(1)^2 + n(1) - 4$

$-11 = m + n - 6$

$0 = m + n + 5 \ \odot$

Solve the system of equations:

$0 = -8m - 2n - 16 \ \odot$

$0 = m + n + 5 \ \odot$

Solve equation \odot for $m$: $m = -n - 5$

Substitute for $m$ in equation $\odot$.

$0 = -8(-n - 5) - 2n - 16$

$m = -1$

$0 = 8n + 40 - 2n - 16$

$0 = 6n + 24$

$n = -4$

So, $m = -1$ and $n = -4$

15. Determine each remainder.

a) $(8x^3 - 6x + 3) \div (4x + 1)$

$$
\begin{array}{c|c}
\text{2x} & -2 \\
\hline
4x & 8x^2 - 6x + 3 \\
\hline
8x^2 & 2x \\
\hline
-8x & 3 \\
\hline
-8x & -2 \\
\hline
5 & \hline
\end{array}
$$

The remainder is 5.

b) $(3x^3 + 2x^2 - 6x - 1) \div (3x + 2)$

$$
\begin{array}{c|c}
x^2 & -2 \\
\hline
3x & 3x^3 + 2x^2 - 6x - 1 \\
\hline
3x^3 & 2x^2 \\
\hline
0 & -6x - 1 \\
-6x & -6x - 4 \\
\hline
3 & \hline
\end{array}
$$

The remainder is 3.