Lesson 3.5 Exercises, pages 232–238

4. Calculate the value of the discriminant for each quadratic equation.

a) $5x^2 - 9x + 4 = 0$
   - In $b^2 - 4ac$, substitute:
     - $a = 5$, $b = -9$, $c = 4$
     - $b^2 - 4ac = (-9)^2 - 4(5)(4) = 1$

b) $3x^2 + 7x - 2 = 0$
   - In $b^2 - 4ac$, substitute:
     - $a = 3$, $b = 7$, $c = -2$
     - $b^2 - 4ac = (7)^2 - 4(3)(-2) = 73$

c) $18x^2 - 12x = 0$
   - In $b^2 - 4ac$, substitute:
     - $a = 18$, $b = -12$, $c = 0$
     - $b^2 - 4ac = (-12)^2 - 4(18)(0) = 144$

d) $6x^2 + 7 = 0$
   - In $b^2 - 4ac$, substitute:
     - $a = 6$, $b = 0$, $c = 7$
     - $b^2 - 4ac = (0)^2 - 4(6)(7) = -168$
5. The values of the discriminant for some quadratic equations are given. How many roots does each equation have?
   a) \( b^2 - 4ac = 36 \)
   The discriminant is positive, so there are 2 real roots.
   
   b) \( b^2 - 4ac = 80 \)
   The discriminant is positive, so there are 2 real roots.
   
   c) \( b^2 - 4ac = 0 \)
   The discriminant is 0, so there is 1 real root.
   
   d) \( b^2 - 4ac = -4 \)
   The discriminant is negative, so there are no real roots.

6. The values of the discriminant for some quadratic equations are given, where \( a, b, \) and \( c \) are integers. In each case, are the roots rational or irrational and can the equation be solved by factoring?
   a) \( b^2 - 4ac = 45 \)
   The square root of the discriminant is irrational, so the roots are irrational and the equation cannot be solved by factoring.
   
   b) \( b^2 - 4ac = -16 \)
   The square root of the discriminant is not a real number, so there are no real roots.
   
   c) \( b^2 - 4ac = 100 \)
   The square root of the discriminant is rational, so the roots are rational and the equation can be solved by factoring.
   
   d) \( b^2 - 4ac = 0 \)
   The discriminant is 0, so the root is rational and the equation can be solved by factoring.

7. Without solving each equation, determine whether it has one, two, or no real roots. Justify your answer.
   a) \( 2x^2 - 9x + 4 = 0 \)
   In \( b^2 - 4ac \), substitute: \( a = 2, \ b = -9, \ c = 4 \)
   \( b^2 - 4ac = (-9)^2 - 4(2)(4) \)
   \( = 49 \)
   Since \( b^2 - 4ac > 0 \), the equation has 2 real roots.
b) \(-x^2 - 7x + 5 = 0\)

In \(b^2 - 4ac\), substitute: \(a = -1, b = -7, c = 5\)

\[b^2 - 4ac = (-7)^2 - 4(-1)(5) = 69\]

Since \(b^2 - 4ac > 0\), the equation has 2 real roots.

c) \(2x^2 + 16x + 32 = 0\)

In \(b^2 - 4ac\), substitute: \(a = 2, b = 16, c = 32\)

\[b^2 - 4ac = 16^2 - 4(2)(32) = 0\]

Since \(b^2 - 4ac = 0\), the equation has 1 real root.

d) \(2.55x^2 - 1.4x - 0.2 = 0\)

In \(b^2 - 4ac\), substitute: \(a = 2.55, b = -1.4, c = -0.2\)

\[b^2 - 4ac = (-1.4)^2 - 4(2.55)(-0.2) = 4\]

Since \(b^2 - 4ac > 0\), the equation has 2 real roots.

8. Determine the values of \(k\) for which each equation has two real roots, then write a possible equation.

a) \(kx^2 + 6x - 1 = 0\)

For an equation to have 2 real roots, \(b^2 - 4ac > 0\)

\[a = k, b = 6, c = -1\]

So, \(6^2 - 4(k)(-1) > 0\)

\[4k > -36\]

\[k > -9\]

Sample response:

\[10x^2 + 6x - 1 = 0\]

b) \(6x^2 - 3x + k = 0\)

For an equation to have 2 real roots, \(b^2 - 4ac > 0\)

\[a = 6, b = -3, c = k\]

So, \((-3)^2 - 4(6)(k) > 0\)

\[24k < 9\]

\[k < \frac{3}{8}\] or \(k > \frac{9}{24}\)

Sample response:

\[6x^2 - 3x - 1 = 0\]

9. Determine the values of \(k\) for which each equation has exactly one real root, then write a possible equation.

a) \(2x^2 - kx + 18 = 0\)

For an equation to have exactly 1 real root, \(b^2 - 4ac = 0\)

\[a = 2, b = -k, c = 18\]

So,

\((-k)^2 - 4(2)(18) = 0\)

\[k^2 = 144\]

\[k = \pm 12\]

Sample response:

\[2x^2 - 12x + 18 = 0\]

or, \(x^2 - 6x + 9 = 0\)

Sample response:

\[\frac{25}{3}x^2 - 10x - 3 = 0\]

or, \(25x^2 + 30x + 9 = 0\)
10. Determine the values of $k$ for which each equation has no real roots, then write a possible equation.

a) $kx^2 - 9x - 3 = 0$

For an equation to have no real roots, $b^2 - 4ac < 0$
Substitute: $a = k, b = -9, c = -3$
$(-9)^2 - 4(k)(-3) < 0$
$12k < -81$
$k < -\frac{81}{12}, \text{ or } -\frac{27}{4}$
Sample response: $-10x^2 - 9x - 3 = 0$

b) $7x^2 - 6x + k = 0$

In $b^2 - 4ac < 0$, substitute: $a = 7, b = -6, c = k$
$(-6)^2 - 4(7)(k) < 0$
$28k > 36$
$k > \frac{36}{28}, \text{ or } \frac{9}{7}$
Sample response: $7x^2 - 6x + 2 = 0$

11. Can each equation be solved by factoring? If your answer is yes, solve it by factoring. If your answer is no, solve it using a different strategy.

a) $7x^2 - 8x - 12 = 0$

An equation factors if its discriminant is a perfect square.
In $b^2 - 4ac$, substitute: $a = 7, b = -8, c = -12$
$b^2 - 4ac = (-8)^2 - 4(7)(-12) = 400$ This is a perfect square.
The equation can be solved by factoring.
$7x^2 - 8x - 12 = 0$
$(7x + 6)(x - 2) = 0$
$x = -\frac{6}{7} \text{ or } x = 2$

b) $14x^2 - 63x - 70 = 0$

Divide by 7.
$2x^2 - 9x - 10 = 0$
In $b^2 - 4ac$, substitute: $a = 2, b = -9, c = -10$
$b^2 - 4ac = (-9)^2 - 4(2)(-10) = 161$ This is not a perfect square.
The equation cannot be solved by factoring.
Substitute: $a = 2, b = -9$ in: $x = \frac{-b \pm \sqrt{161}}{2a}$
$x = \frac{9 \pm \sqrt{161}}{2(2)}$
$x = \frac{9 \pm \sqrt{161}}{4}$
12. For each equation below:

i) Determine the value of the discriminant.

ii) Use the value of the discriminant to choose a solution strategy, then solve the equation.

a) \(2x^2 - 6x + 1 = 0\) 
   
i) In \(b^2 - 4ac\), substitute:
   
   \[a = 2, \ b = -6, \ c = 1\]
   
   \[b^2 - 4ac = (-6)^2 - 4(2)(1) = 28\]
   
   This is not a perfect square, so use the quadratic formula.
   
   \[x = \frac{-b \pm \sqrt{28}}{2a}\]
   
   \[x = \frac{6 \pm \sqrt{28}}{2(2)}\]
   
   \[x = \frac{3 \pm \sqrt{7}}{2}\]

b) \(8x^2 - 3x - 5 = 0\) 

   i) In \(b^2 - 4ac\), substitute:
   
   \[a = 8, \ b = -3, \ c = -5\]
   
   \[b^2 - 4ac = (-3)^2 - 4(8)(-5) = 169\]
   
   This is a perfect square, so use factoring.
   
   \[(8x + 5)(x - 1) = 0\]
   
   \[x = -\frac{5}{8} \text{ or } x = 1\]

13. A model rocket is launched. Its height, \(h\) metres, after \(t\) seconds is described by the formula \(h = 23t - 4.9t^2\). Without solving an equation, determine whether the rocket reaches each height.

a) 20 m 
   
   In \(h = 23t - 4.9t^2\), substitute:
   
   \[h = 20\]
   
   \[20 = 23t - 4.9t^2\]
   
   If the rocket reaches a height of 20 m, then the equation has real roots.
   
   In \(b^2 - 4ac\), substitute:
   
   \[a = -4.9, \ b = 23, \ c = -20\]
   
   \[b^2 - 4ac = 23^2 - 4(-4.9)(-20) = 137\]
   
   Since the discriminant is positive, the equation has real roots, and the rocket reaches a height of 20 m.

b) 30 m 
   
   In \(h = 23t - 4.9t^2\), substitute:
   
   \[h = 30\]
   
   \[30 = 23t - 4.9t^2\]
   
   If the rocket reaches a height of 30 m, then the equation has real roots.
   
   In \(b^2 - 4ac\), substitute:
   
   \[a = -4.9, \ b = 23, \ c = -30\]
   
   \[b^2 - 4ac = 23^2 - 4(-4.9)(-30) = -59\]
   
   Since the discriminant is negative, the equation has no real roots, and the rocket does not reach a height of 30 m.
14. Create three different quadratic equations whose discriminant is 64. Explain your strategy. What is true about these three equations?

Sample response:
Use guess and test to determine 3 values of $a$, $b$, and $c$ so that:

$b^2 - 4ac = 64$ Substitute: $b = 0$
Then $-4ac = 64$, which is satisfied by $a = -4$ and $c = 4$
So, one equation is: $-4x^2 + 4 = 0$

$b^2 - 4ac = 64$ Substitute: $b = 4$
Then $16 - 4ac = 64$, and $-4ac = 48$, which is satisfied by $a = -3$ and $c = 4$
So, another equation is: $-3x^2 + 4x + 4 = 0$

$b^2 - 4ac = 64$ Substitute: $b = 2$
Then $4 - 4ac = 64$, and $-4ac = 60$, which is satisfied by $a = 5$ and $c = -3$
So, another equation is: $5x^2 + 2x - 3 = 0$

All 3 equations have 2 rational real roots.

15. Consider the equation $5x^2 + 6x + k = 0$. Determine two positive values of $k$ for which this equation has two rational roots.

To be able to factor, the discriminant must be a perfect square.
In $b^2 - 4ac$, substitute: $a = 5$, $b = 6$, $c = k$

$6^2 - 4(5)(k) = 36 - 20k$

Use guess and test.

One perfect square is 16.

Another perfect square is 4.

36 - 20k = 16
20k = 20
$k = 1$
36 - 20k = 4
20k = 32
$k = \frac{32}{20}$, or 1.6

16. Create a quadratic equation so that $a$, $b$, and $c$ are real numbers; the value of the discriminant is a perfect square; but the roots are not rational. Justify your solution by determining the value of the discriminant and the roots of the equation.

Sample response: The equation has the form $ax^2 + bx + c = 0$
Consider the quadratic formula:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If the roots are not rational, then $a$ or $b$ is irrational.
Use guess and test. Suppose $b = \sqrt{5}$.
Then $b^2 - 4ac$ must be a perfect square.
Substitute: $b = \sqrt{5}$

$5 - 4ac$ must be a perfect square, such as 25.

$5 - 4ac = 25$

$4ac = -20$

$ac = -5$, which is satisfied by $a = 1$, $c = -5$
An equation is: $x^2 + \sqrt{5}x - 5 = 0$
The discriminant is 25.
The roots are: $x = \frac{-\sqrt{5} \pm 5}{2}$
17. Consider the quadratic formula: \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

a) Write expressions for the two roots of the quadratic equation \( ax^2 + bx + c = 0 \).

The two roots are: 
\[ x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

b) Add the expressions. How is this sum related to the coefficients of the quadratic equation?

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a}, \text{ or } -\frac{b}{a} \]

The sum of the roots is the opposite of the quotient of the coefficient of \( x \) and the coefficient of \( x^2 \).

c) Multiply the expressions. How is this product related to the coefficients of the quadratic equation?

\[ \left( \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right) \left( \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) = \frac{b^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} \]
\[ = \frac{4ac}{4a^2}, \text{ or } \frac{c}{a} \]

The product of the roots is the quotient of the constant term and the coefficient of \( x^2 \).

d) Use the results from parts b and c to write an equation whose roots are \( x = -3 \pm \sqrt{11} \).

\[ \frac{-b}{a} = (-3 + \sqrt{11}) + (-3 - \sqrt{11}) \]
\[ \frac{-b}{a} = -6, \text{ or } \frac{-6}{1} \]
\[ \frac{c}{a} = (-3 + \sqrt{11})(-3 - \sqrt{11}) \]
\[ = 9 - 11 \]
\[ \frac{c}{a} = -2, \text{ or } \frac{-2}{1} \]

So, \( a = 1, b = 6, \text{ and } c = -2 \)

An equation is: \( x^2 + 6x - 2 = 0 \)